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The Concept of Proper Solution in Linear Programming[1]

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Abstract. In this paper, we study the optimal solutions of a dual pair of linear programming problems that correspond to the proper equilibria of their associated matrix game. We give conditions ensuring the existence of such solutions, show that they are especially robust under perturbation of right-hand-side terms, and describe a procedure to obtain them.

Key Words. Linear programming, game theory, proper equilibria.

1. Introduction

A significant issue in the theory of linear programming is whether solutions are stable under perturbation of the data of the problem. Williams (Ref. 1) and Robinson (Ref. 2) gave necessary and sufficient conditions for the stability of the primal and dual solution sets of solvable linear programming problems; Wendell (Ref. 3) defined the tolerance of right-hand-side terms and objective function coefficients as the maximum percentage change that does not change the optimal basis.

In spite of the well-known relationship between the optimal solutions of a dual pair of linear programming problems and the Nash equilibria of an associated matrix game (see Refs. 4–5), we know of no previous attempt

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to choose between alternative optima in linear programming by making use of refinements of the concept of Nash equilibrium that have appeared in the last two decades in response to the recognition that not all Nash equilibria are equally self-enforcing; for an exhaustive survey, see Ref. 6.

In this paper, we discuss the significance for linear programming of the proper equilibria defined by Myerson (Ref. 7), which for matrix games constitute the nucleolus of the game (Ref. 8) and are the equilibria played by players aiming to maximize the minimum gain that is obtained if the opponent makes a mistake (Ref. 6). The paper is organized as follows. In Section 2, we establish notation, recall known results on matrix games and linear programming, and prove two new results concerning their relationship. In Section 3, we define the proper solutions of a dual pair of linear programming problems as the solutions corresponding to the proper equilibria of the corresponding matrix game, give sufficient conditions for the existence of such solutions, and use the properties of the nucleolus of a matrix game (Refs. 8–9) to show that proper solutions are especially robust in that they are the optimal solutions saturating the least possible number of constraints. Finally, in the appendices, we describe a procedure for calculating the proper solutions of a dual pair of linear programming problems and illustrate the procedure with an example.

2. Matrix Games and Linear Programming

An $m \times n$ matrix game Γ is defined by $\Gamma = (\Delta_1, \Delta_2, A)$, where Δ_1 $[\Delta_2]$, the simplex of $\mathbb{R}^m[\mathbb{R}^n]$, is the set of mixed strategies of player 1 [2] and A is an $m \times n$ matrix defining the payoff functions K_i of the players as follows: for all $(\xi, \eta) \in \Delta_1 \times \Delta_2$,

$$K_1(\xi, \eta) = \xi^t A \eta \quad \text{and} \quad K_2(\xi, \eta) = -\xi^t A \eta.$$

We denote by S_1 and S_2 the canonical basis of $\mathbb{R}^m$ and $\mathbb{R}^n$ (i.e., the sets of pure strategies of the players) and by e_s and f_s the sth elements of S_1 and S_2, respectively. The value $v(\Gamma)$ of the game Γ is defined by

$$v(\Gamma) = \max_{\xi \in \Delta_1} \min_{\eta \in \Delta_2} \xi^t A \eta = \min_{\eta \in \Delta_2} \max_{\xi \in \Delta_1} \xi^t A \eta.$$

The set of optimal strategies of player 1 is given by

$$O_1(\Gamma) = \{\xi \in \Delta_1 | \xi^t A f_j \geq v(\Gamma), j = 1, \ldots, n\},$$

and the set of optimal strategies of player 2 by

$$O_2(\Gamma) = \{\eta \in \Delta_2 | e_i^t A \eta \leq v(\Gamma), i = 1, \ldots, m\}.$$

A pair $(\xi, \eta) \in \Delta_1 \times \Delta_2$ is a Nash equilibrium of Γ if and only if $(\xi, \eta) \in O_1(\Gamma) \times O_2(\Gamma)$. The strategies $\xi_1, \xi_2 \in \Delta_1$ are said to be payoff-equivalent strategies for player 1 if and only if

$$K_1(\xi_1, \eta) = K_1(\xi_2, \eta), \text{ for all } \eta \in \Delta_2;$$

payoff-equivalent strategies for player 2 are defined analogously. An $n \times n$ matrix game $\Gamma = (\Delta_1, \Delta_2, A)$ is called a symmetric game if and only if

$$A = -A^t.$$

For a symmetric $n \times n$ matrix game Γ,

$$O_1(\Gamma) = O_2(\Gamma) \quad \text{and} \quad v(\Gamma) = 0;$$

for proof, see for example Ref. 4; since $O_1(\Gamma) = O_2(\Gamma)$, we can drop the subscripts and write $O(\Gamma)$ for both.

The nucleolus $N(\Pi, F)$ of a nonempty convex set $\Pi \subset \mathbb{R}^s$ and a continuous convex map $F: \Pi \to \mathbb{R}^l$ is the set

$$\{x \in \Pi \mid \theta \circ F(x) \succcurlyeq_{\text{lex}} \theta \circ F(y), \text{ for all } y \in \Pi\},$$

where the map $\theta: \mathbb{R}^l \to \mathbb{R}^l$ orders the coordinates of any point $z \in \mathbb{R}^l$ in nondecreasing order; the lexicographic ordering $\succcurlyeq_{\text{lex}}$ is defined on $\mathbb{R}^l$ by

$x \succcurlyeq_{\text{lex}} y$ if and only if $x = y$ or there exists $k \leq l$ such that

$x_i = y_i$ if $i < k$ and $x_k > y_k$.

Given an $m \times n$ matrix game $\Gamma = (\Delta_1, \Delta_2, A)$, we define

$$N_I(\Gamma) = N(\Delta_1, F) \quad \text{and} \quad N_{II}(\Gamma) = N(\Delta_2, G),$$

where $F: \Delta_1 \to \mathbb{R}^n$ is such that

$$F_j(\xi) = \xi^t A f_j$$

and $G: \Delta_2 \to \mathbb{R}^m$ is such that

$$G_i(\eta) = -e_i^t A \eta.$$

The nucleolus of the matrix game Γ is defined by

$$N(\Gamma) = N_I(\Gamma) \times N_{II}(\Gamma).$$

A dual pair (P, D) of linear programming problems in canonical form is defined by

$$\begin{aligned} \text{(P)} \quad \max \quad & x^t c, \\ \text{s.t.} \quad & x^t A \leq b^t, \\ & x \geq 0, \end{aligned}$$

and

$$\text{(D)}\quad \min\quad y^t b,\qquad \text{s.t.}\quad Ay \geq c,\quad y \geq 0.$$

Here, y, $b \in \mathbb{R}^n$; x, $c \in \mathbb{R}^m$; and A is an $m \times n$ matrix. If both P and D have nonempty sets of optimal solutions, we call them finite dual linear programming problems. With every pair (P, D), finite or not, we associate the symmetric $(m+n+1) \times (m+n+1)$ matrix game Γ defined by the matrix

$$B = \begin{bmatrix} 0 & -A & c \\ A^t & 0 & -b \\ -c^t & b^t & 0 \end{bmatrix}.$$

Theorem 2.1. Let P and D be a dual pair of linear programming problems, and let $\Gamma = (\Delta_1, \Delta_2, B)$ be their associated symmetric game.

(i) If (X, Y) is a pair of optimal solutions of (P, D), then

$$\xi^t = (X^t/\lambda,\ Y^t/\lambda,\ 1/\lambda), \quad \text{with} \quad \lambda = \sum_{i=1}^{m} X_i + \sum_{j=1}^{n} Y_j + 1,$$

is an optimal strategy of Γ.

(ii) If $\xi^t = (x^t, y^t, a) \in \Delta_1$ is an optimal strategy of Γ, with $a > 0$, then

$$X = x/a,\ Y = y/a$$

are optimal solutions of P and D, respectively.

(iii) If all the optimal strategies of Γ are of the form $\xi^t = (x^t, y^t, 0)$, then neither P nor D has any optimal solution.

Proof. See Ref. 4. □

The following theorem relates optimal strategies of the form $\xi^t = (x^t, y^t, 0)$ to the boundedness of the optimal solution sets of (P, D) when both these sets are nonempty.

Theorem 2.2. Let P and D be a pair of finite dual linear programming problems, and let $\Gamma = (\Delta_1, \Delta_2, B)$ be their associated symmetric game.

(i) If Γ has an optimal strategy of the form $(x^t, y^t, 0)$, then:

(a) $x^t c = 0 = y^t b$,

(b) either the optimal solution set of P or the optimal solution set of D is unbounded, or both.

(ii) If the optimal solution set of P or the optimal solution set of D is unbounded, then Γ has a pure strategy that is optimal, whose last coordinate is zero.

Proof.

(i) Let $\xi^t = (x^t, y^t, 0)$ be an optimal strategy of Γ. Since the value of Γ is zero,

$$y^t A^t \geq 0, \; -x^t A \geq 0, \; x^t c - y^t b \geq 0.$$

Moreover, since P and D both have optimal solutions [(P, D) being a finite pair], there is an optimal strategy $\bar{\xi}^t = (\bar{x}^t, \bar{y}^t, \bar{\lambda})$ with $\bar{\lambda} > 0$ such that $\bar{x}/\bar{\lambda}$ and $\bar{y}/\bar{\lambda}$ are optimal solutions of P and D, respectively. For $\alpha > 0$ the strategy

$$\zeta^t = [(\bar{x}^t + \alpha x^t)/(1+\alpha), \; (\bar{y}^t + \alpha y^t)/(1+\alpha), \; \bar{\lambda}/(1+\alpha)]$$

is also an optimal strategy; since $\bar{\lambda}/(1+\alpha) > 0$, then $(\bar{x}^t + \alpha x^t)/\bar{\lambda}$ and $(\bar{y}^t + \alpha y^t)/\bar{\lambda}$ are optimal solutions of P and D, respectively. Hence,

$$\bar{x}^t c/\bar{\lambda} + \alpha x^t c/\bar{\lambda} = \bar{x}^t c/\bar{\lambda} \quad \text{and} \quad \bar{y}^t b/\bar{\lambda} + \alpha y^t b/\bar{\lambda} = \bar{y}^t b/\bar{\lambda},$$

which implies that

$$x^t c = 0 = y^t b.$$

Now either $x \neq 0$ or $y \neq 0$. Suppose that $x \neq 0$. Since $\bar{x}/\bar{\lambda}$ is an optimal solution of P and $-x^t A \geq 0$, then for $\alpha > 0$,

$$-\tilde{x}^t A + b = -\bar{x}^t A/\bar{\lambda} + b - \alpha x^t A \geq 0,$$

where

$$\tilde{x} = \bar{x}/\bar{\lambda} + \alpha x.$$

Moreover, since $x^t c = 0$, then

$$\tilde{x}^t c = (\bar{x}^t/\bar{\lambda} + \alpha x^t) c = (\bar{x}^t/\bar{\lambda}) c;$$

since $\tilde{x}$ is feasible (note that $\tilde{x} \geq 0$), $\tilde{x}$ is an optimal solution of P. Since $x \neq 0$, increasing α makes the nonzero coordinates of $\tilde{x}$ as large as desired. Thus, the set of optimal solutions of P is unbounded.

Analogously, if $y \neq 0$, the set of optimal solutions of D is unbounded.

(ii) Suppose that the set of optimal solutions of P is unbounded, and let $i \in \{1, \ldots, m\}$ be an unbounded coordinate. Let X and Y be a pair of optimal solutions of P and D, respectively, and let

$$\xi^t = (X^t/\lambda, \; Y^t/\lambda, \; 1/\lambda), \text{ with } \lambda = \sum_{l=1}^{m} X_l + \sum_{j=1}^{n} Y_j + 1,$$

the associated optimal strategy of Γ. If we take a sequence of optimal solutions $\{(X^k, Y)\}_{k \in \mathbb{N}}$ such that the ith coordinate of X^k tends to infinity, then the corresponding sequence of optimal strategies converges to $\zeta = e_i$, which is thus an optimal strategy of Γ that is pure and has a zero final coordinate.

The same result follows if the set of optimal solutions of D is unbounded. □

The final result in this section states the relationship between the slacks of the optimal solutions of a dual pair of finite linear programming problems and the optimal strategies of their associated symmetric game.

Lemma 2.1. Let P and D be a dual pair of finite linear programming problems, and let $\Gamma = (\Delta_1, \Delta_2, B)$ be their associated symmetric game. Let X and Y be a pair of optimal solutions of P and D, respectively, and let, $\xi^t = (X^t/\lambda, Y^t/\lambda, 1/\lambda)$ be the corresponding optimal strategy of Γ, where $\lambda = \sum_{i=1}^m X_i + \sum_{j=1}^n Y_j + 1$. Let $h^{\mathrm{P}}(X)$ and $h^{\mathrm{D}}(Y)$ be the vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$ comprising the slacks of X and Y, respectively. Then:

(i) $\xi^t B f_j = (1/\lambda) h_j^{\mathrm{D}}(Y)$, for all $j = 1, \ldots, m$;
(ii) $\xi^t B f_j = (1/\lambda) h_{j-m}^{\mathrm{P}}(X)$, for all $j = m+1, \ldots, m+n$;
(iii) $\xi^t B f_j = 0$, if $j = m+n+1$.

Proof. Take $j \in \{1, \ldots, m\}$. Then,

$$\xi^t B f_j = (Y^t/\lambda) A^t f_j - (1/\lambda) c_j = (1/\lambda)(Y^t A^t f_j - c_j) = (1/\lambda) h_j^{\mathrm{D}}(Y),$$

by the definition of the vector $h^{\mathrm{D}}(Y)$. Similarly, if $j \in \{m+1, \ldots, m+n\}$,

$$\begin{aligned} \xi^t B f_j &= -(X^t/\lambda) A f_j + (1/\lambda) b_{j-m} \\ &= (1/\lambda)(-X^t A f_j + b_{j-m}) \\ &= (1/\lambda) h_{j-m}^{\mathrm{P}}(X), \end{aligned}$$

by the definition of the vector $h^{\mathrm{P}}(X)$. Finally, when $j = m+n+1$,

$$\xi^t B f_j = (X^t/\lambda) c - (Y^t/\lambda) b = 0,$$

because X and Y are optimal solutions of P and D, respectively. □

3. Proper Solutions in Linear Programming

Let $\Gamma = (\Delta_1, \Delta_2, A)$ be an $m \times n$ matrix game. A pair of strategies (ξ, η) is a proper equilibrium of Γ if and only if there exist sequences $\{\epsilon_k\}_{k \in \mathbb{N}}$ and

$\{(\xi_k, \eta_k)\}_{k\in\mathbb{N}}$ such that:

(i) for all $k\in\mathbb{N}$, $\epsilon_k > 0$, $(\xi_k)_i > 0$, for all $i = 1, \ldots, m$, and $(\eta_k)_s > 0$, for all $s = 1, \ldots, n$;

(ii) for all $k\in\mathbb{N}$, given $e_i, e_j \in S_1$, with $e_i^t A\eta_k < e_j^t A\eta_k$, then $(\xi_k)_i \leq \epsilon_k(\xi_k)_j$;

(iii) for all $k\in\mathbb{N}$, given $f_r, f_s \in S_2$, with $\xi_k^t Af_r < \xi_k^t Af_s$, then $(\eta_k)_s \leq \epsilon_k(\eta_k)_r$;

(iv) $\lim_{k\to\infty} \epsilon_k = 0$ and $\lim_{k\to\infty}(\xi_k, \eta_k) = (\xi, \eta)$.

We define the set

$$PR_1(\Gamma) = \{\xi\in\Delta_1 | \exists\eta\in\Delta_2 \text{ such that } (\xi, \eta) \text{ is a proper equilibrium of } \Gamma\},$$

and with obvious changes, the set $PR_2(\Gamma)$.

Theorem 3.1. For any symmetric matrix game Γ, $PR_1(\Gamma) = PR_2(\Gamma)$.

Proof. Let (ξ, η) be a proper equilibrium of Γ by virtue of the sequences $\{\epsilon_k\}_{k\in\mathbb{N}}$ and $\{(\xi_k, \eta_k)\}_{k\in\mathbb{N}}$. We show that the sequences $\{\epsilon_k\}_{k\in\mathbb{N}}$ and $\{(\eta_k, \xi_k)\}_{k\in\mathbb{N}}$ define the proper equilibrium (η, ξ).

Take $k\in\mathbb{N}$. Let e_i, e_j be pure strategies of player 1 such that

$$e_i^t A\xi_k < e_j^t A\xi_k.$$

Since $A = -A^t$,

$$\xi_k^t Ae_j < \xi_k^t Ae_i.$$

Thus,

$$(\eta_k)_i \leq \epsilon_k(\eta_k)_j.$$

Similarly, for $f_r, f_s \in S_2$,

$$(\xi_k)_s \leq \epsilon_k(\xi_k)_r, \text{ if } \eta_k^t Af_s > \eta_k^t Af_r.$$

This finishes the proof of the theorem. □

By virtue of Theorem 3.1, for symmetric matrix games, we can drop the subscripts in $PR_1(\Gamma)$ and $PR_2(\Gamma)$.

Definition 3.1. Let P and D be a dual pair of finite linear programming problems, and let Γ be their associated symmetric game. The optimal solutions X and Y are called the proper solutions of P and D, respectively, if and only if

$$(X^t/\lambda,\ Y^t/\lambda,\ 1/\lambda)\in PR(\Gamma), \text{ where } \lambda = \sum_{i=1}^{m} X_i + \sum_{j=1}^{n} Y_j + 1.$$

The following example shows that there are finite linear programming problems with no proper solutions.

Example 3.1. Consider the following pair of linear programming problems:

$$\text{(P1)} \quad \max \quad 0x,$$
$$\text{s.t.} \quad -2x \leq 1,$$
$$x \geq 0,$$

and

$$\text{(D1)} \quad \min \quad y,$$
$$\text{s.t.} \quad -2y \geq 0,$$
$$y \geq 0.$$

The optimal solution set of Problem (P1) is $[0, \infty)$ and that of Problem (D1) is $\{0\}$.

The associated symmetric game is defined by the matrix

$$B = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Its set of Nash equilibria is $\mathrm{Conv}\{e_1, e_3\} \times \mathrm{Conv}\{e_1, e_3\}$. Since the pure strategy e_3 is dominated, this game has only one proper equilibrium (e_1, e_1), so

$$PR(\Gamma) = \{e_1\}.$$

Since the final coordinate of e_1 is zero, it cannot correspond to any proper solution of (P1, D1), by Definition 3.1. Hence (P1, D1) has no proper solution.

Theorem 3.2. Let P and D be a dual pair of finite linear programming problems, and let $\Gamma = (\Delta_1, \Delta_2, B)$ be their associated symmetric game. The pair (P, D) has at least one proper solution if either (i), or (ii), or (iii) below holds:

(i) both P and D have bounded optimal solution sets;
(ii) one of the optimal solution sets is unbounded, but all the Nash equilibria of Γ are payoff-equivalent;
(iii) $\xi^t B = 0$, for all $\xi \in \Omega = \{(x^t, y^t, 0) \mid (x^t, y^t, 0) \in O(\Gamma)\}$.

Proof.

(i) If P and D have bounded optimal solution sets, then the result follows immediately from Theorem 2.2.

(ii) If P, D and Γ satisfy condition (ii), then all the Nash equilibria are proper. Since P and D are finite, by Theorem 2.1 there exists at least one optimal strategy (x^t, y^t, a) with $a > 0$. The solution corresponding to this strategy is a proper solution.

(iii) If $\xi^t B = 0$, for all $\xi \in \Omega$, then either all the Nash equilibria of Γ are payoff-equivalent, in which case there exists as above at least one proper solution, or the elements of Ω are dominated strategies. In this latter situation, since all proper equilibria are undominated, the strategies involved in each proper equilibrium are elements of $\Delta_1 \backslash \Omega$. The solutions associated with these strategies are proper solutions of P and D. □

The converse of this result is not true, as the following example shows.

Example 3.2. Consider the dual finite linear programming problems

$$\text{(P2)} \quad \max \quad 0x,$$
$$\text{s.t.} \quad -2x \leq 3,$$
$$x \geq 0,$$

and

$$\text{(D2)} \quad \min \quad 3y,$$
$$\text{s.t.} \quad -2y \geq 0,$$
$$y \geq 0.$$

The optimal solution set of problem (P2) is $[0, \infty)$ and that of problem (D2) is $\{0\}$.

The associated symmetric game Γ is given by

$$B = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}.$$

Its set of Nash equilibria is $\text{Conv}\{e_1, e_3\} \times \text{Conv}\{e_1, e_3\}$, $\xi^t B \geq 0$ for all $\xi \in O(\Gamma)$, and not all the $\xi \in O(\Gamma)$ are payoff-equivalent. This game has a unique proper equilibrium, (e_3, e_3), because the pure strategy e_1 is dominated. Thus, $(X = 0, Y = 0)$ is a proper solution of (P2, D2).

For our main result, we need the following theorem, whose proof is given in Ref. 8.

Theorem 3.3. The nucleolus of a matrix game is the set of its proper equilibria.

Our main result now follows.

Theorem 3.4. Let P and D be a dual pair of finite linear programming problems. Let $(\tilde{X}, \tilde{Y})$ be a proper solution of (P, D), and let $\tilde{\lambda} = \sum_{i=1}^{m} \tilde{X}_i + \sum_{j=1}^{n} \tilde{Y}_j + 1$. Let X and Y be any pair of optimal solutions of P and D, respectively, and let $\lambda = \sum_{i=1}^{m} X_i + \sum_{j=1}^{n} Y_j + 1$. Then,

$$\theta((1/\tilde{\lambda})h(\tilde{X}, \tilde{Y})) \geqslant_{\text{lex}} \theta((1/\lambda)h(X, Y)),$$

where

$$h(X,Y) = \begin{bmatrix} h^{\text{P}}(X) \\ h^{\text{D}}(Y) \end{bmatrix}, \qquad h(\tilde{X}, \tilde{Y}) = \begin{bmatrix} h^{\text{P}}(\tilde{X}) \\ h^{\text{D}}(\tilde{Y}) \end{bmatrix}.$$

Moreover,

$$(1/\lambda)h(X, Y) = (1/\tilde{\lambda})h(\tilde{X}, \tilde{Y}),$$

if and only if (X, Y) is a proper solution of (P, D).

Proof. Let P and D be a dual pair of finite linear programming problems, and let $\Gamma = (\Delta_1, \Delta_2, B)$ be their associated symmetric game. By the definition of proper solutions,

$$\tilde{\xi}^t = (\tilde{X}^t/\tilde{\lambda}, \tilde{Y}^t/\tilde{\lambda}, 1/\tilde{\lambda}) \in PR(\Gamma).$$

Therefore, by Theorem 3.3,

$$\theta(\tilde{\xi}^t B) \geqslant_{\text{lex}} \theta(\xi^t B), \text{ for all } \xi \in \Delta_1.$$

Thus, for

$$\xi^t = (X^t/\lambda, Y^t/\lambda, 1/\lambda),$$

Lemma 2.1 shows that

$$\theta((1/\tilde{\lambda})h(\tilde{X}, \tilde{Y})) \geqslant_{\text{lex}} \theta((1/\lambda)h(X, Y)).$$

If (X, Y) is a proper solution of (P, D), then

$$\xi^t = (X^t/\lambda, Y^t/\lambda, 1/\lambda) \in PR(\Gamma) = N_I(\Gamma),$$

and so

$$(1/\lambda)h(X,Y) = (1/\tilde{\lambda})h(\tilde{X},\tilde{Y}).$$

Conversely, if

$$(1/\lambda)h(X, Y) = (1/\tilde{\lambda})h(\tilde{X}, \tilde{Y}),$$

then

$$\zeta^t = (X^t/\lambda, Y^t/\lambda, 1/\lambda) \in N_I(\Gamma) = PR(\Gamma),$$

so (X, Y) is a proper solution of (P, D). □

Theorem 3.4 shows that proper solutions are the solutions saturating the least number of constraints in P and D. In other words, they are optimal solutions that are especially robust in that they remain stable under modification of the largest number of right-hand sides.

4. Appendix A: Solution-Finding Procedure

The set of proper equilibria of a matrix game can be found by means of the Dresher procedure (Ref. 6). A modified form of this procedure is used in the following algorithm to find the proper solutions of a dual pair of finite linear programming problems.

Input and Initialization. The input is the following dual pair of finite linear programming problems:

$$\text{(P)} \quad \max \quad x^t c, \qquad \text{s.t.} \quad x^t A \leq b^t, \quad x \geq 0,$$

and

$$\text{(D)} \quad \min \quad y^t b, \qquad \text{s.t.} \quad Ay \geq c, \quad y \geq 0.$$

Set $k = 0$ and

$$B^0 = \begin{bmatrix} 0 & -A & c \\ A^t & 0 & -b \\ -c^t & b^t & 0 \end{bmatrix}.$$

Algorithmic Steps.

Step 1. Consider the matrix game Γ^k with matrix B^k. Let P^k be the following problem:

$$(\mathrm{P}^k) \quad \max \quad v,$$
$$\text{s.t.} \quad \xi^t B^k \geq v e^t,$$
$$\xi^t e = 1,$$
$$\xi \geq 0,$$

where e is the vector with coordinates all equal to unity. Let M^k be the set of optimal solutions of P^k. The set of optimal strategies of Γ^k is

$$O(\Gamma^k) = \{\xi \,|\, (\xi, v) \in M^k\}.$$

Step 2. Let S^{k+1} be the set of the extreme points of $O(\Gamma^k)$. We define the set of indices

$$N^{k+1} = \{j \,|\, \text{there exists } (\xi, v) \in M^k \text{ such that } \xi^t B^k f_j > v\}.$$

Step 3. If $N^{k+1} = \emptyset$, go to Step 5. Otherwise, consider the game Γ^{k+1} with matrix $B^{k+1} = (b_{ij}^{k+1})$ such that

$$b_{ij}^{k+1} = \xi_i^t B^k f_j, \text{ for all } \xi_i \in S^{k+1} \text{ and } j \in N^{k+1}.$$

Step 4. Set $k = k + 1$, and go to Step 1.

Step 5. The extreme points of $PR(\Gamma^0)$ are the elements of S^{k+1} expressed with respect to the canonical basis (see the following example and Ref. 6 on the Dresher procedure).

If the last coordinate of any $\xi \in PR(\Gamma^0)$ is zero, (P, D) has no proper solution. Every $\xi^t = (x^t, y^t, \lambda) \in PR(\Gamma^0)$ with $\lambda > 0$ gives a proper solution $(x/\lambda, y/\lambda)$ of (P, D).

5. Appendix B: Example

Consider the following dual pair of linear programming problems:

$$(\mathrm{P3}) \quad \max \quad x_1 + x_2,$$
$$\text{s.t.} \quad 2x_1 + 2x_2 \leq 10,$$
$$x_1 + x_2 + x_3 \leq 5,$$
$$-x_1 \leq -1,$$
$$2x_1 - 2x_2 \leq 2,$$
$$x_1,\ x_2 \geq 0,$$

and

$$\begin{aligned}
\text{(D3)}\quad \min\quad & 10y_1 + 5y_2 - y_3 + 2y_4,\\
\text{s.t.}\quad & 2y_1 + y_2 - y_3 + 2y_4 \geq 1,\\
& 2y_1 + y_2 - 2y_4 \geq 1,\\
& y_1, y_2, y_3, y_4 \geq 0.
\end{aligned}$$

For these problems, the matrix B^0 is given by

$$B^0 = \begin{bmatrix} 0 & 0 & 0 & -2 & -1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & 0 & -10 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & -5 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & -2 \\ -1 & -1 & 0 & 10 & 5 & -1 & 2 & 0 \end{bmatrix},$$

and problem P^0 is

$$\begin{aligned}
(\mathrm{P}^0)\quad \max\quad & v,\\
\text{s.t.}\quad & \xi^t B^0 \geq v e^t,\\
& \xi^t e = 1,\\
& \xi \geq 0,
\end{aligned}$$

where e^t is the row vector whose coordinates are all equal to 1. The set of extreme points of the optimal strategies of Γ^0 is

$$S^1 = \left\{ \begin{bmatrix} 2/13 \\ 8/13 \\ 0 \\ 1/13 \\ 0 \\ 0 \\ 0 \\ 2/13 \end{bmatrix}, \begin{bmatrix} 6/13 \\ 4/13 \\ 0 \\ 1/13 \\ 0 \\ 0 \\ 0 \\ 2/13 \end{bmatrix}, \begin{bmatrix} 1/7 \\ 4/7 \\ 0 \\ 0 \\ 1/7 \\ 0 \\ 0 \\ 1/7 \end{bmatrix}, \begin{bmatrix} 3/7 \\ 2/7 \\ 0 \\ 0 \\ 1/7 \\ 0 \\ 0 \\ 1/7 \end{bmatrix} \right\},$$

and $N^1 = \{3, 6, 7\}$.

Since $N^1 \neq \emptyset$, we increment k to $k=1$. Now,

$$B^1 = \begin{bmatrix} 0 & 0 & 16/13 \\ 0 & 4/13 & 0 \\ 1/7 & 0 & 8/7 \\ 1/7 & 2/7 & 0 \end{bmatrix},$$

problem P^1 is

$$(\mathrm{P}^1) \quad \max \quad v,$$
$$\text{s.t.} \quad \xi^t B^1 \geq v e^t,$$
$$\xi^t e = 1,$$
$$\xi \geq 0,$$

and

$$S^2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/8 \\ 7/8 \end{bmatrix} \right\}, \; N^2 = \{2, 3\}.$$

Since $N^2 \neq \emptyset$, we increment k to $k=2$. In this step,

$$B^2 = \begin{bmatrix} 1/7 & 4/7 \\ 1/4 & 1/7 \end{bmatrix},$$

$$S^3 = \left\{ \begin{bmatrix} 1/5 \\ 4/5 \end{bmatrix} \right\}, \; N^3 = \emptyset.$$

The set

$$PR(\Gamma^0) = \{(13/35, 12/35, 0, 0, 1/7, 0, 0, 1/7)\}$$

is obtained as

$$((1/2)(1/5) + (4/5)(1/8))(1/7, 4/7, 0, 0, 1/7, 0, 0, 1/7)$$
$$+ ((1/2)(1/5) + (4/5)(7/8))(3/7, 2/7, 0, 0, 1/7, 0, 0, 1/7).$$

Hence, the original problems have a proper solution given by

$$X^* = (13/5, 12/5, 0), \qquad Y^* = (0, 1, 0, 0).$$

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